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Positive solutions for three-point nonlinear fractional boundary value problems*

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Abstract

In this paper, we give sufficient conditions for the existence or the nonexistence of positive solutions of the nonlinear fractional boundary value problem

$$D_{0+}^{\alpha}u + a(t)f(u(t)) = 0,\ 0 < t < 1,\ 2 < \alpha < 3,$$
$$u(0) = u'(0) = 0,\ \ u'(1) - \mu u'(\eta) = \lambda,$$

where D_{0+}^{α} is the standard Riemann-Liouville fractional differential operator of order α, $\eta \in (0,1)$, $\mu \in \left[0, \dfrac{1}{\eta^{\alpha-2}}\right)$ are two arbitrary constants and $\lambda \in [0,\infty)$ is a parameter. The proof uses the Guo-Krasnosel'skii fixed point theorem and Schauder's fixed point theorem.

1 Introduction

In this paper, we are interested in the existence or non-existence of positive solutions for the nonlinear fractional boundary value problem (BVP for short)

$$D_{0+}^{\alpha}u + a(t)f(u(t)) = 0,\ 2 < \alpha < 3,\ 0 < t < 1, \tag{1.1}$$

$$u(0) = u'(0) = 0,\ u'(1) - \mu u'(\eta) = \lambda, \tag{1.2}$$

where α is a real number, D_{0+}^{α} is the standard Riemann-Liouville differentiation of order α, $\eta \in (0,1)$, $\mu \in \left[0, \dfrac{1}{\eta^{\alpha-2}}\right)$ are arbitrary constants and $\lambda \in [0,\infty)$ is a parameter. A positive solution is a function $u(t)$ which is positive on $(0,1)$ and satisfies (1.1)-(1.2).

We show, under suitable conditions on the nonlinear term f, that the fractional boundary value problem (1.1)-(1.2) has at least one or has non positive solutions. By employing the fixed point theorems for operators acting on cones

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in a Banach space (see, for example [7, 8, 13, 14, 15]). The use of cone techniques in order to study boundary value problems has a rich and diverse history. That is, some authors have used fixed point theorems to show the existence of positive solutions to boundary value problems for ordinary differential equations, difference equations, and dynamic equations on time scales, (see for example [1, 2, 3]). Moreover, Delbosco and Rodino [7] considered the existence of a solution for the nonlinear fractional differential equation $D_{0+}^{\alpha} u = f(t,u)$, where $0 < \alpha < 1$ and $f : [0,a] \times \mathbb{R} \to \mathbb{R}$, $0 < a \leq +\infty$ is a given function, continuous in $(0,a) \times \mathbb{R}$. They obtained results for solutions by using the Schauder fixed point theorem and the Banach contraction principle. Bai and Lü [5] studied the existence and multiplicity of positive solutions of nonlinear fractional differential equation boundary value problem:

$$\begin{cases} D_{0+}^{\alpha} u + f(t,u(t)) = 0, \;\; 1 < \alpha < 2, \, 0 < t < 1, \\ u(0) = u(1) = 0, \end{cases}$$

where D_{0+}^{α} is the standard Riemann-Liouville differential operator of order α. Recently Bai and Qiu [4]. considered the existence of positive solutions to boundary value problems of the nonlinear fractional differential equation

$$\begin{cases} D_{0+}^{\alpha} u + f(t,u(t)) = 0, \;\; 2 < \alpha \leq 2, \, 0 < t < 1, \\ u(0) = u'(1) = u''(0) = 0, \end{cases}$$

where D_{0+}^{α} is the Caputo's fractional differentiation, and $f : (0,1] \times [0,+\infty) \to [0,+\infty)$, with $\lim_{t \longrightarrow 0^+} f(t,.) = +\infty$. They obtained results for solutions by using the Krasnoselskii's fixed point theorem and the nonlinear alternative of Leray-Schauder type in a cone.

Lü Zhang [18] considered the existence of solutions of nonlinear fractional boundary value problem involving Caputo's derivative

$$\begin{cases} D_t^{\alpha} u + f(t,u(t)) = 0, \, 1 < \alpha < 2, \, 0 < t < 1, \\ u(0) = \nu \neq 0, \quad u(1) = \rho \neq 0. \end{cases}$$

In another paper, by using fixed point theory on cones, Zhang [19] studied the existence and multiplicity of positive solution of nonlinear fractional boundary value problem

$$\begin{cases} D_t^{\alpha} u + f(t,u(t)) = 0, \, 1 < \alpha < 2, \, 0 < t < 1, \\ u(0) + u'(0) = 0, \, u(1) + u'(1) = 0, \end{cases}$$

where D_t^{α} is the Caputo's fractional derivative. By using the Krasnoselskii fixed point theory on cones, Benchohra, Henderson, Ntoyuas and Ouahab [6] used the Banach fixed point and the nonlinear alternative of Leray-Schauder to investigate the existence of solutions for fractional order functional and neutral functional differential equations with infinite delay

$$\begin{cases} D^{\alpha} y(t) = f(t, y_t), \quad \text{for each } t \in J = [0,b] \, , \, 0 < \alpha < 1, \\ y(t) = \phi(t), \quad t \in (-\infty,0], \end{cases}$$

where D^{α} is the standard Riemman-Liouville fractional derivative, $f : J \times B \rightarrow \mathbb{R}$ is a given function satisfying some suitable assumptions, $\phi \in B$, $\phi(0) = 0$ and B is called a phase space. By using the Krasnoselskii fixed point theory on cones, El-Shahed [8] Studied the existence and nonexistence of positive solutions to nonlinear fractional boundary value problem

$$\begin{cases} D_{0^+}^{\alpha} u + \lambda a(t) f(u(t)) = 0, \ 2 < \alpha < 3, \ 0 < t < 1, \\ \qquad u(0) = u'(0) = u'(1) = 0, \end{cases}$$

where $D_{0^+}^{\alpha}$ is the standard Riemann-Liouville differential operator of order α. Some existence results were given for the problem (1.1)-(1.2) with $\alpha = 3$ by Sun [17].The BVP (1.1)-(1.2) arises in many different areas of applied mathematics and physics, and only its positive solution is significant in some practice.

For existence theorems of fractional differential equation and application, the definitions of fractional integral and derivative and related proprieties we refer the reader to [7, 11, 12, 16].

The rest of this paper is organized as follows: In section 2, we present some preliminaries and lemmas. Section 3 is devoted to prove the existence and nonexistence of positive solutions for BVP (1.1)-(1.2).

2 Elementary Background and Preliminary lemmas

In this section, we will give the necessary notations, definitions and basic lemmas that will be used in the proofs of our main results. We also present a fixed point theorem due to Guo and Krasnosel'skii.

Definition 1 *[10, 11, 15]. The fractional (arbitrary) order integral of the function $h \in \mathbb{L}^1([a,b], \mathbb{R}_+)$ of order $\alpha \in \mathbb{R}_+$ is defined by*

$$I_a^{\alpha} h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} h(s) ds,$$

where Γ is the gamma function. When $a = 0$, we write $I^{\alpha} h(t) = (h * \varphi_{\alpha})(t)$, where $\varphi_{\alpha}(t) = \dfrac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t > 0$, and $\varphi_{\alpha}(t) = 0$ for $t \leq 0$, and $\varphi_{\alpha} \longrightarrow \delta(t)$ as $\alpha \longrightarrow 0$, where δ is the delta function.

Definition 2 *[10, 11, 15]. For a function h given on the interval $[a,b]$, the αth Riemann-Liouville fractional-order derivative of h, is defined by*

$$\left(D_{a^+}^{\alpha} h\right)(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{f(s)}{(t-s)^{\alpha-n+1}} ds, \quad n = [\alpha] + 1.$$

Lemma 3 *[4] Let $\alpha > 0$. If $u \in C(0,1) \cap L(0,1)$, then the fractional differential equation*

$$D_{0^+}^{\alpha} u(t) = 0 \tag{2.2}$$

has solution $u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + ... + c_n t^{\alpha-n}$, $c_i \in \mathbb{R}$, $i = 1, 2, ..., n$, $n = [\alpha] + 1$.

Lemma 4 *[4] Assume that* $u \in C(0,1) \cap L(0,1)$ *with a fractional derivative of order* $\alpha > 0$. *Then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + ... + c_n t^{\alpha-n}, \tag{2.3}$$

for some $c_i \in \mathbb{R}$, $i = 1, 2, ..., n$, $n = [\alpha] + 1$.

Lemma 5 *Let* $y \in C^+[0,1] = \{y \in C[0,1], \, y(t) \geq 0, \, t \in [0,1] \, \}$, *then the (BVP)*

$$D_{0+}^{\alpha} u(t) + y(t) = 0, \ \ 2 < \alpha < 3, \ 0 < t < 1, \tag{2.4}$$

$$u(0) = u'(0) = 0, \ \ u'(1) - \mu u'(\eta) = \lambda, \tag{2.5}$$

has a unique solution

$$u(t) = \int_0^1 G(t,s) y(s) ds + \tfrac{\mu t^{\alpha-1}}{(1-\mu\eta^{\alpha-2})} \int_0^1 G_1(\eta, s) y(s) ds + \tfrac{\lambda t^{\alpha-1}}{2(1-\mu\eta)} \tag{2.6}$$

where

$$G(t,s) = \tfrac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-2} - (t-s)^{\alpha-1}, & s \leq t \\ t^{\alpha-1}(1-s)^{\alpha-2}, & t \leq s \end{cases} \tag{2.7}$$

and

$$G_1(\eta, s) = \tfrac{1}{\Gamma(\alpha)} \begin{cases} \eta^{\alpha-2}(1-s)^{\alpha-2} - (\eta-s)^{\alpha-2}, & s \leq \eta \\ \eta^{\alpha-2}(1-s)^{\alpha-2}, & \eta \leq s \end{cases} \tag{2.8}$$

Proof. By applying Lemmas 3 and 4 , the equation (2.4) is equivalent to the following integral equation

$$u(t) = -c_1 t^{\alpha-1} - c_2 t^{\alpha-2} - c_3 t^{\alpha-3} - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds. \tag{2.9}$$

for some arbitrary constants $c_1, c_2, c_3 \in \mathbb{R}$. Boundary conditions (2.5), permit us to deduce there exacts values

$$c_2 = c_3 = 0$$

$$c_1 = \tfrac{1}{(\mu\eta^{\alpha-2}-1)} \left\{ \tfrac{1}{\Gamma(\alpha)} \left[\int_0^1 (1-s)^{\alpha-2} y(s) ds - \mu \int_0^{\eta} (\eta-s)^{\alpha-2} y(s) ds \right] + \tfrac{\lambda}{(\alpha-1)} \right\}$$

then, the unique solution of (2.4)-(2.5) is given by the formula

$$\begin{aligned} u(t) &= -\tfrac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds + \tfrac{t^{\alpha-1}}{(1-\mu\eta^{\alpha-2})\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-2} y(s) ds \\ &\quad - \tfrac{\mu t^{\alpha-1}}{(1-\mu\eta^{\alpha-2})\Gamma(\alpha)} \int_0^{\eta} (\eta-s)^{\alpha-2} y(s) ds + \tfrac{\lambda t^{\alpha-1}}{(\alpha-1)(1-\mu\eta^{\alpha-2})} \\ &= -\tfrac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds + \tfrac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-2} y(s) ds \end{aligned}$$

$$+\tfrac{\mu\eta^{\alpha-2}t^{\alpha-1}}{(1-\mu\eta^{\alpha-2})\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-2}y(s)ds-\tfrac{\mu t^{\alpha-1}}{(1-\mu\eta^{\alpha-2})\Gamma(\alpha)}\int_0^\eta(\eta-s)^{\alpha-2}y(s)ds$$
$$+\tfrac{\lambda t^{\alpha-1}}{(\alpha-1)(1-\mu\eta^{\alpha-2})}$$
$$=-\tfrac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}y(s)ds+\tfrac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^t(1-s)^{\alpha-2}y(s)ds$$
$$+\tfrac{t^{\alpha-1}}{\Gamma(\alpha)}\int_t^1(1-s)^{\alpha-2}y(s)ds+\tfrac{\mu t^{\alpha-1}\eta^{\alpha-2}}{(1-\mu\eta^{\alpha-2})\Gamma(\alpha)}\int_0^\eta(1-s)^{\alpha-2}y(s)ds$$
$$+\tfrac{\mu t^{\alpha-1}\eta^{\alpha-2}}{(1-\mu\eta^{\alpha-2})\Gamma(\alpha)}\int_\eta^1(1-s)^{\alpha-2}y(s)ds$$
$$+\frac{-1}{\eta^{\alpha-2}}\tfrac{\mu t^{\alpha-1}\eta^{\alpha-2}}{(1-\mu\eta^{\alpha-2})\Gamma(\alpha)}\int_0^\eta(\eta-s)^{\alpha-2}y(s)ds+\tfrac{\lambda t^{\alpha-1}}{(\alpha-1)(1-\mu\eta^{\alpha-2})}$$
$$=\tfrac{1}{\Gamma(\alpha)}\int_0^1G(t,s)y(s)ds+\tfrac{\mu t^{\alpha-1}}{(1-\mu\eta^{\alpha-2})\Gamma(\alpha)}\int_0^1G_1(\eta,s)y(s)ds+\tfrac{\lambda t^{\alpha-1}}{(\alpha-1)(1-\mu\eta^{\alpha-2})}$$

where,

$$G(t,s)=\frac{1}{\Gamma(\alpha)}\begin{cases} t^{\alpha-1}(1-s)^{\alpha-2}-(t-s)^{\alpha-1}\ , & s\le t\\ t^{\alpha-1}(1-s)^{\alpha-2}, & t\le s\end{cases}$$

$$G_1(\eta,s)=\frac{1}{\Gamma(\alpha)}\begin{cases} \eta^{\alpha-2}(1-s)^{\alpha-2}-(\eta-s)^{\alpha-2}, & s\le\eta\\ \eta^{\alpha-2}(1-s)^{\alpha-2}\ , & \eta\le s\end{cases}$$

This ends the proof. In order to check the existence of positive solutions, we give some properties of the functions $G(t,s)$ and $G_1(t,s)$. ■

Lemma 6 *For all* $(t,s)\in[0,1]\times[0,1]$*, we have*

(P1) $\dfrac{\partial G(t,s)}{\partial t}=(\alpha-1)\,G_1(t,s)$.

(P2) $0\le G_1(\eta,s)\le\frac{1}{\Gamma(\alpha)}\eta^{\alpha-2}\,(1-s)^{\alpha-2}$, $\int_0^1G_1(\eta,s)ds=\frac{(1-\eta)\eta^{\alpha-2}}{(\alpha-1)\Gamma(\alpha)}$

(P3) $\gamma G(1,s)\le G(t,s)\le G(1,s)$, $(t,s)\in[\tau,1]\times[0,1]$.

Where $G(1,s)=\frac{1}{\Gamma(\alpha)}s(1-s)^{\alpha-2}$, $\gamma=\tau^{\alpha-2}$, and τ satisfies

$$\int_\tau^1 s(1-s)^{\alpha-2}a(s)ds>0. \tag{2.10}$$

Proof. (P1) and (P2) are obvious. We prove that (P3) holds.

For all $(t,s)\in[0,1]\times[0,1]$, (P1) and (P2) imply that, $0\le G(t,s)\le G(1,s)$.

If $0\le s\le t\le 1$, we have

$$\begin{aligned}\frac{G(t,s)}{G(1,s)}&=\frac{t^{\alpha-1}(1-s)^{\alpha-2}-(t-s)^{\alpha-1}}{s(1-s)^{\alpha-2}}\\ &\ge\frac{t(t-ts)^{\alpha-2}-(t-ts)^{\alpha-1}}{s(1-s)^{\alpha-2}}=\frac{t(t-ts)^{\alpha-2}-(t-ts)(t-ts)^{\alpha-2}}{s(1-s)^{\alpha-2}}\\ &=\frac{ts(t-ts)^{\alpha-2}}{s(1-s)^{\alpha-2}}=t^{\alpha-1}.\end{aligned}$$

If $0 \leq t \leq s \leq 1$, we have

$$\frac{G(t,s)}{G(1,s)} = \frac{t^{\alpha-1}(1-s)^{\alpha-2}}{s(1-s)^{\alpha-2}} \geq \frac{t^{\alpha-1}(1-s)^{\alpha-2}}{(1-s)^{\alpha-2}} = t^{\alpha-1}.$$

Thus,

$$t^{\alpha-1}G(1,s) \leq G(t,s) \leq G(1,s),\ (t,s) \in [0,1] \times [0,1].$$

Therefore,

$$\tau^{\alpha-1}G(1,s) \leq G(t,s) \leq G(1,s),\ (t,s) \in [\tau,1] \times [0,1].$$

This completes the proof. ■

Lemma 7 *If $y \in C^+[0,1]$, then the unique solution $u(t)$ of the BVP (2.4)-(2.5) is nonnegative and satisfies*

$$\min_{t\in[\tau,1]} u(t) \geq \gamma \|u\|.$$

Proof. Let $y \in C^+[0,1]$; it is obvious that $u(t)$ is nonnegative. For any $t \in [0,1]$, by (2.6) and Lemma 6, it follows that

$$\begin{aligned} u(t) &= \int_0^1 G(t,s)y(s)ds + \tfrac{\mu t^{\alpha-1}}{(1-\mu\eta^{\alpha-2})}\int_0^1 G_1(\eta,s)y(s)ds + \tfrac{\lambda t^{\alpha-1}}{(\alpha-1)(1-\mu\eta^{\alpha-2})} \\ &\leq \int_0^1 G(1,s)y(s)ds + \tfrac{\mu}{(1-\mu\eta^{\alpha-2})}\int_0^1 G_1(\eta,s)y(s)ds + \tfrac{\lambda}{(\alpha-1)(1-\mu\eta^{\alpha-2})}, \end{aligned}$$

and thus

$$\|u\| \leq \int_0^1 G(1,s)y(s)ds + \tfrac{\mu}{(1-\mu\eta^{\alpha-2})}\int_0^1 G_1(\eta,s)y(s)ds + \tfrac{\lambda}{(\alpha-1)(1-\mu\eta^{\alpha-2})}.$$

More that, (2.6) and Lemma 6 imply that, for any $t \in [\tau,1]$,

$$\begin{aligned} u(t) &= \int_0^1 G(t,s)y(s)ds + \tfrac{\mu t^{\alpha-1}}{(1-\mu\eta^{\alpha-2})}\int_0^1 G_1(\eta,s)y(s)ds + \tfrac{\lambda t^{\alpha-1}}{(\alpha-1)(1-\mu\eta^{\alpha-2})} \\ &\geq \gamma\int_0^1 G(1,s)y(s)ds + \tfrac{\mu\tau^{\alpha-1}}{(1-\mu\eta^{\alpha-2})}\int_0^1 G_1(\eta,s)y(s)ds + \tfrac{\lambda\tau^{\alpha-1}}{(\alpha-1)(1-\mu\eta^{\alpha-2})} \\ &= \gamma\left(\int_0^1 G(1,s)y(s)ds + \tfrac{\mu}{(1-\mu\eta^{\alpha-2})}\int_0^1 G_1(\eta,s)y(s)ds + \tfrac{\lambda}{(\alpha-1)(1-\mu\eta^{\alpha-2})}\right). \end{aligned}$$

Hence

$$\min_{t\in[\tau,1]} u(t) \geq \gamma \|u\|.$$

This completes the proof. ■

Definition 8 *Let $\mathcal{E}$ be a real Banach space. A nonempty closed convex set $\mathcal{K} \subset \mathcal{E}$ is called cone of $\mathcal{E}$ if it satisfies the following conditions*

(A1) $x\in\mathcal{K}$, $\sigma\geq 0$ implies $\sigma x\in\mathcal{K}$;
(A2) $x\in\mathcal{K}$, $-x\in\mathcal{K}$ implies $x=0$.

Definition 9 *An operator is called completely continuous if it continuous and maps bounded sets into precompact sets*

To establish the existence or nonexistence of positive solutions of BVP (1.1)-(1.2), we will employ the following Guo-Krasnosel'skii fixed point theorem:

Theorem 10 *[12] Let $\mathcal{E}$ be a Banach space and let $\mathcal{K}\subset\mathcal{E}$ be a cone in $\mathcal{E}$. Assume that Ω_1 and Ω_2 are open subsets of $\mathcal{E}$ with $0\in\Omega_1$ and $\overline{\Omega}_1\subset\Omega_2$. Let $T:\mathcal{K}\cap\left(\overline{\Omega}_2\backslash\Omega_1\right)\longrightarrow\mathcal{K}$ be completely continuous operator. In addition, suppose either*

(H1) $\|Tu\|\leq\|u\|$, $\forall u\in\mathcal{K}\cap\partial\Omega_1$ and $\|Tu\|\geq\|u\|$, $\forall u\in\mathcal{K}\cap\partial\Omega_2$ or
(H2) $\|Tu\|\leq\|u\|$, $\forall u\in\mathcal{K}\cap\partial\Omega_2$ and $\|Tu\|\geq\|u\|$, $\forall u\in\mathcal{K}\cap\partial\Omega_1$,
holds. Then T has a fixed point in $\mathcal{K}\cap\left(\overline{\Omega}_2\backslash\Omega_1\right)$.

3 Existence of solutions

In this section, we will apply Krasnosel'skii's fixed point theorem to the problem (1.1)-(1.2). We note that $u(t)$ is a solution of (1.1)-(1.2) if and only if

$$\begin{aligned} u(t) = \int_0^1 G(t,s)a(s)f(u(s))ds + \tfrac{\mu t^{\alpha-1}}{(1-\mu\eta^{\alpha-2})}\int_0^1 G_1(\eta,s)a(s)f(u(s))ds \\ +\tfrac{\lambda t^{\alpha-1}}{(\alpha-1)(1-\mu\eta^{\alpha-2})}. \end{aligned} \tag{3.1}$$

Let us consider the Banach space of the form

$$\mathcal{E}=C^{+}\left[0,1\right]=\left\{u\in C\left[0,1\right],\, u(t)\geq 0,\, t\in[0,1]\ \right\},$$

equipped with standard norm

$$\|u\|_{\infty}=\max\left\{|u(t)|:t\in[0,1]\right\}.$$

We define a cone $\mathcal{K}$ by

$$\mathcal{K}=\left\{u\in\mathcal{E}:\min_{t\in[\tau,1]}u(t)\geq\gamma\|u\|\right\},$$

and an integral operator $T:\mathcal{E}\longrightarrow\mathcal{E}$ by

$$\begin{aligned} Tu(t) = \int_0^1 G(t,s)a(s)f(u(s))ds + \tfrac{\mu t^{\alpha-1}}{(1-\mu\eta^{\alpha-2})}\int_0^1 G_1(\eta,s)a(s)f(u(s))ds \\ +\tfrac{\lambda t^{\alpha-1}}{(\alpha-1)(1-\mu\eta^{\alpha-2})}. \end{aligned} \tag{3.2}$$

It is not difficult see that, fixed points of T are solutions of (1.1)-(1.2). Our aim is to show that $T:\mathcal{K}\longrightarrow\mathcal{K}$ is completely continuous, in order to use Theorem 10.

Lemma 11 *Let $f : [0, \infty) \longrightarrow [0, \infty)$ continuous. Assume the following condition*

(C0) $a \in C([0, 1], [0, \infty))$.

Then operator $T : \mathcal{K} \longrightarrow \mathcal{K}$ is completely continuous.

Proof. Since $G(t, s)$, $G_1(\eta, s) \geq 0$, then $Tu(t) \geq 0$ for all $u \in \mathcal{K}$. We first prove that $T(\mathcal{K}) \subset \mathcal{K}$. In fact,

$$\begin{aligned} Tu(t) &= \int_0^1 G(t, s)a(s)f(u(s))ds + \tfrac{t^{\alpha-1}}{(1-\mu\eta^{\alpha-2})} \left\{\mu \int_0^1 G_1(\eta, s)a(s)f(u(s))ds + \tfrac{\lambda}{(\alpha-1)}\right\} \\ &\leq \int_0^1 G(1, s)a(s)f(u(s))ds + \tfrac{1}{(1-\mu\eta^{\alpha-2})} \left\{\mu \int_0^1 G_1(\eta, s)a(s)f(u(s))ds + \tfrac{\lambda}{(\alpha-1)}\right\} \\ &t \in [0, 1] \end{aligned}$$

so,

$$\|Tu\| \leq \int_0^1 G(1, s)a(s)f(u(s))ds + \tfrac{1}{(1-\mu\eta^{\alpha-2})} \left\{\mu \int_0^1 G_1(\eta, s)a(s)f(u(s))ds + \tfrac{\lambda}{(\alpha-1)}\right\},$$

on the other hand, Lemma 6 imply that, for any $t \in [\tau, 1]$,

$$\begin{aligned} Tu(t) &= \int_0^1 G(t, s)a(s)f(u(s))ds + \tfrac{t^{\alpha-1}}{(1-\mu\eta^{\alpha-2})} \left\{\mu \int_0^1 G_1(\eta, s)a(s)f(u(s))ds + \tfrac{\lambda}{(\alpha-1)}\right\} \\ &\geq \gamma \int_0^1 G(1, s)a(s)f(u(s))ds + \tfrac{\tau^{\alpha-1}}{(1-\mu\eta^{\alpha-2})} \left\{\mu \int_0^1 G_1(\eta, s)a(s)f(u(s))ds + \tfrac{\lambda}{(\alpha-1)}\right\} \\ &= \gamma \int_0^1 G(1, s)a(s)f(u(s))ds + \tfrac{\gamma}{(1-\mu\eta^{\alpha-2})} \left(\mu \int_0^1 G_1(\eta, s)a(s)f(u(s))ds + \tfrac{\lambda}{(\alpha-1)}\right) \end{aligned}$$

and, for $u \in \mathcal{K}$

$$\min_{t \in [\tau, 1]} Tu(t) \geq \gamma \|Tu\|.$$

Consequently, we have $T(\mathcal{K}) \subset \mathcal{K}$. Next, we prove that T is continuous. In fact, let

$$N = \frac{1}{2\Gamma(\alpha)} \left(\int_0^1 s(1-s)^{\alpha-2}a(s)ds + \int_0^1 (1-s)^{\alpha-2}a(s)ds\right),$$

assume that u_n, $u_0 \in \mathcal{K}$ and $u_n \longrightarrow u_0$, then $\|u_n\| \leq c < \infty$, for every $n \geq 0$. Since f is continuous on $[0, c]$, it is uniformly continuous. Therefore, for any $\epsilon > 0$, there exists $\delta > 0$ such that $|u_1 - u_2| < \delta$ implies that $|f(u_1) - f(u_2)| < \frac{\varepsilon}{2N}$. Since $u_n \longrightarrow u_0$, there exists $n_0 \in \mathbb{N}$ such that $\|u_n - u_0\| < \delta$ for $n \geq n_0$. Thus we have $|f(u_n(t)) - f(u_0(t))| < \frac{\varepsilon}{2N}$, for $n \geq n_0$and $t \in [0, 1]$. This implies that for $n \geq n_0$

$$
\begin{aligned}
\|Tu_n - Tu_0\| &= \left\| \begin{array}{c} \int_0^1 G(t,s)a(s)\left[f(u_n(s)) - f(u_0(s))\right] ds \\ +\frac{\mu t^{\alpha-1}}{(1-\mu\eta^{\alpha-2})}\int_0^1 G_1(\eta,s)a(s)\left[f(u_n(s)) - f(u_0(s))\right] ds \end{array} \right\| \\
&\leq \frac{\varepsilon}{2N}\left[\int_0^1 G(1,s)a(s)ds + \frac{\mu}{(1-\mu\eta^{\alpha-2})}\int_0^1 G_1(\eta,s)a(s)ds\right] \\
&\leq \frac{\varepsilon}{2N}\left[\frac{1}{\Gamma(\alpha)}\int_0^1 s(1-s)^{\alpha-2}a(s)ds + \frac{\mu\eta^{\alpha-2}}{(1-\mu\eta^{\alpha-2})}\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-2}a(s)ds\right] \\
&\leq \frac{\varepsilon}{2N}\left[\frac{1}{\Gamma(\alpha)}\left(\int_0^1 s(1-s)^{\alpha-2}a(s)ds + \frac{1}{(1-\mu\eta^{\alpha-2})}\int_0^1 (1-s)^{\alpha-2}a(s)ds\right)\right] \\
&\leq \frac{\varepsilon}{2N}(2N) = \varepsilon.
\end{aligned}
$$

That is, $T : \mathcal{K} \longrightarrow \mathcal{K}$ is continuous. Finally, let $\mathcal{B} \subset \mathcal{K}$ be bounded, we claim that $T(\mathcal{B}) \subset \mathcal{K}$ is uniformly bounded. Indeed, since $\mathcal{B}$ is bounded, there exists some $m > 0$ such that $\|u\| \leq m$, for all $u \in \mathcal{B}$. Let

$$C = \max\{|f(u(t))| : 0 \leq u \leq m\}$$

then

$$\|Tu\| \leq C_1 N \quad \text{for all } u \in \mathcal{B}.$$

such that $C_1 = C + \frac{\lambda}{(\alpha-1)(1-\mu\eta^{\alpha-2})N}$ At last, we prove $T(\mathcal{B})$ is equicontinuous. Hence $T(\mathcal{B})$ is bounded, for all $\varepsilon > 0$, each $u \in \mathcal{B}$, $t_1, t_2 \in [0,1]$, $t_1 < t_2$, let $\delta = \min\left\{\frac{\Gamma(\alpha)\varepsilon}{6C\|a\|_\infty}, \frac{\left(1-\mu\eta^{\alpha-2}\right)\Gamma(\alpha)\varepsilon}{3C\|a\|_\infty}, \frac{\left(1-\mu\eta^{\alpha-2}\right)\varepsilon}{3\lambda}\right\}$, this allows us to show that,

$$|Tu(t_2) - Tu(t_1)| < \varepsilon \text{ when } t_2 - t_1 < \delta.$$

One has

$$
\begin{aligned}
&|Tu(t_2) - Tu(t_1)| \\
&= \left| \begin{array}{c} \int_0^1 \left(G(t_2,s) - G(t_1,s)\right) a(s)f(u(s))ds \\ +\frac{\left(t_2^{\alpha-1} - t_1^{\alpha-1}\right)}{(1-\mu\eta^{\alpha-2})}\left\{\mu\int_0^1 G_1(\eta,s)a(s)f(u(s))ds + \frac{\lambda}{(\alpha-1)}\right\} \end{array} \right| \\
&\leq C\|a\|_\infty \left(\int_0^1 \left(G(t_2,s) - G(t_1,s)\right) ds + \frac{\mu\left(t_2^{\alpha-1} - t_1^{\alpha-1}\right)}{(1-\mu\eta^{\alpha-2})}\int_0^1 G_1(\eta,s)ds\right) \\
&+ \frac{\lambda\left(t_2^{\alpha-1} - t_1^{\alpha-1}\right)}{(\alpha-1)(1-\mu\eta^{\alpha-2})} \\
&\leq C\|a\|_\infty \left\{\int_0^{t_1} \left(G(t_2,s) - G(t_1,s)\right) ds + \int_{t_1}^{t_2} \left(G(t_2,s) - G(t_1,s)\right) ds\right\} \\
&+ C\|a\|_\infty \int_{t_2}^1 \left(G(t_2,s) - G(t_1,s)\right) ds \\
&+ \frac{\mu\left(t_2^{\alpha-1} - t_1^{\alpha-1}\right)}{(1-\mu\eta^{\alpha-2})}\frac{C\|a\|_\infty}{\Gamma(\alpha)}\int_0^1 G_1(\eta,s)ds + \frac{\lambda\left(t_2^{\alpha-1} - t_1^{\alpha-1}\right)}{(\alpha-1)(1-\mu\eta^{\alpha-2})}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{C\left\|a\right\|_{\infty}}{\Gamma(\alpha)}\left(I_1+I_2+I_3\right)+\tfrac{\left(t_2^{\alpha-1}-t_1^{\alpha-1}\right)}{(1-\mu\eta^{\alpha-2})}\left(C\left\|a\right\|_{\infty}\mu\textstyle\int_0^1 G_1(\eta,s)ds+\tfrac{\lambda}{(\alpha-1)}\right)\\
&= \frac{C\left\|a\right\|_{\infty}}{\Gamma(\alpha)}\left(I_4+I_5+I_6\right)+\tfrac{\left(t_2^{\alpha-1}-t_1^{\alpha-1}\right)}{(1-\mu\eta^{\alpha-2})}\left(C\left\|a\right\|_{\infty}\mu\textstyle\int_0^1 G_1(\eta,s)ds+\tfrac{\lambda}{(\alpha-1)}\right)\\
&= \frac{C\left\|a\right\|_{\infty}}{\Gamma(\alpha)}\left(I_7+I_8+I_9\right)+\tfrac{\left(t_2^{\alpha-1}-t_1^{\alpha-1}\right)}{(1-\mu\eta^{\alpha-2})}\left(C\left\|a\right\|_{\infty}\mu\textstyle\int_0^1 G_1(\eta,s)ds+\tfrac{\lambda}{(\alpha-1)}\right)\\
&= \frac{C\left\|a\right\|_{\infty}}{\Gamma(\alpha)}\left(I_{10}+I_{11}+I_{12}\right)+\tfrac{\left(t_2^{\alpha-1}-t_1^{\alpha-1}\right)}{(1-\mu\eta^{\alpha-2})}\left(C\left\|a\right\|_{\infty}\mu\textstyle\int_0^1 G_1(\eta,s)ds+\tfrac{\lambda}{(\alpha-1)}\right)\\
&= \frac{C\left\|a\right\|_{\infty}}{\Gamma(\alpha)}I_{13}+\tfrac{\mu\left(t_2^{\alpha-1}-t_1^{\alpha-1}\right)}{(1-\mu\eta^{\alpha-2})}C\left\|a\right\|_{\infty}\frac{(1-\eta)\,\eta^{\alpha-2}}{(\alpha-1)\,\Gamma(\alpha)}+\tfrac{\lambda\left(t_2^{\alpha-1}-t_1^{\alpha-1}\right)}{(\alpha-1)(1-\mu\eta^{\alpha-2})}\\
&\leq \frac{C\left\|a\right\|_{\infty}}{\Gamma(\alpha)}I_{13}+\tfrac{\mu\left(t_2^{\alpha-1}-t_1^{\alpha-1}\right)}{(1-\mu\eta^{\alpha-2})}\frac{C\left\|a\right\|_{\infty}}{\Gamma(\alpha)}\frac{\eta^{\alpha-2}}{\alpha-1}+\tfrac{\lambda\left(t_2^{\alpha-1}-t_1^{\alpha-1}\right)}{(\alpha-1)(1-\mu\eta^{\alpha-2})},
\end{aligned}
$$

where

$I_1=\int_0^{t_1}\left[(1-s)^{\alpha-2}(t_2^{\alpha-1}-t_1^{\alpha-1})-\left((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right)\right]ds$

$I_2=\int_{t_1}^{t_2}\left[(1-s)^{\alpha-2}(t_2^{\alpha-1}-t_1^{\alpha-1})-(t_2-s)^{\alpha-1}\right]ds$

$I_3=\int_{t_2}^{1}(1-s)^{\alpha-2}(t_2^{\alpha-1}-t_1^{\alpha-1})ds$

$I_4=(t_2^{\alpha-1}-t_1^{\alpha-1})\int_0^{t_1}(1-s)^{\alpha-2}ds-\int_0^{t_1}(t_2-s)^{\alpha-1}ds$

$I_5=(t_2^{\alpha-1}-t_1^{\alpha-1})\int_0^{t_1}(1-s)^{\alpha-2}ds-\int_0^{t_1}(t_2-s)^{\alpha-1}ds$

$I_6=-\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds+(t_2^{\alpha-1}-t_1^{\alpha-1})\int_{t_2}^{1}(1-s)^{\alpha-2}ds$

$I_7=\frac{1}{\alpha-1}(t_2^{\alpha-1}-t_1^{\alpha-1})\left[1-(1-t_1)^{\alpha-1}\right]+\frac{1}{\alpha}\left[(t_2-t_1)^{\alpha}-{t_2}^{\alpha}\right]+\frac{1}{\alpha}{t_1}^{\alpha}$

$I_8=-\frac{1}{\alpha-1}(t_2^{\alpha-1}-t_1^{\alpha-1})\left[(1-t_2)^{\alpha-1}-(1-t_1)^{\alpha-1}\right]$

$I_9=-\frac{1}{\alpha}(t_2-t_1)^{\alpha}+\frac{1}{\alpha-1}(t_2^{\alpha-1}-t_1^{\alpha-1})(1-t_2)^{\alpha-1}$

$I_{10}=\frac{1}{\alpha-1}(t_2^{\alpha-1}-t_1^{\alpha-1})-\frac{1}{\alpha-1}(t_2^{\alpha-1}-t_1^{\alpha-1})(1-t_1)^{\alpha-1}$

$I_{11}=\int\frac{1}{\alpha}(t_2-t_1)^{\alpha}-\frac{1}{\alpha}{t_2}^{\alpha}+\frac{1}{\alpha}{t_1}^{\alpha}-\frac{1}{\alpha-1}(t_2^{\alpha-1}-t_1^{\alpha-1})(1-t_2)^{\alpha-1}$

$I_{12}=\frac{1}{\alpha-1}(t_2^{\alpha-1}-t_1^{\alpha-1})(1-t_1)^{\alpha-1}-\frac{1}{\alpha}(t_2-t_1)^{\alpha}+\frac{1}{\alpha-1}(t_2^{\alpha-1}-t_1^{\alpha-1})(1-t_2)^{\alpha-1}$

$I_{13}=\frac{1}{\alpha-1}(t_2^{\alpha-1}-t_1^{\alpha-1})-\frac{1}{\alpha}(t_2^{\alpha}-t_1^{\alpha}).$

In order to estimate ${t_2}^{\alpha}-{t_1}^{\alpha}$ and ${t_2}^{\alpha-1}-{t_1}^{\alpha-1}$, we can apply a method used in [4, 18]; by means value theorem of differentiation, we have

$$
\begin{gathered}
{t_2}^{\alpha}-{t_1}^{\alpha}\leq\alpha(t_2-t_1)<\alpha\delta\leq 3\delta,\\
{t_2}^{\alpha-1}-{t_1}^{\alpha-1}\leq(\alpha-1)\,(t_2-t_1)<(\alpha-1)\,\delta\leq 2\delta.
\end{gathered}
$$

Thus, we obtain

$$
\begin{aligned}
&\left|Tu(t_2)-Tu(t_1)\right|<\frac{C\left\|a\right\|_{\infty}}{\Gamma(\alpha)}I+\tfrac{\mu\left(t_2^{\alpha-1}-t_1^{\alpha-1}\right)}{(1-\mu\eta^{\alpha-2})}\frac{C\left\|a\right\|_{\infty}}{\Gamma(\alpha)}\frac{\eta^{\alpha-2}}{\alpha-1}+\tfrac{\lambda\left(t_2^{\alpha-1}-t_1^{\alpha-1}\right)}{(\alpha-1)(1-\mu\eta^{\alpha-2})}\\
&<\frac{C\left\|a\right\|_{\infty}}{\Gamma(\alpha)}\left[\frac{(\alpha-1)\,\delta}{(\alpha-1)}+\frac{\alpha\delta}{\alpha}\right]+\tfrac{\mu(\alpha-1)\delta}{(1-\mu\eta^{\alpha-2})}\frac{C\left\|a\right\|_{\infty}}{\Gamma(\alpha)}\frac{\eta^{\alpha-2}}{(\alpha-1)}+\tfrac{\lambda(\alpha-1)\delta}{(\alpha-1)(1-\mu\eta^{\alpha-2})}\\
&<\left(\frac{2C\left\|a\right\|_{\infty}}{\Gamma(\alpha)}+\tfrac{1}{(1-\mu\eta^{\alpha-2})}\frac{C\left\|a\right\|_{\infty}}{\Gamma(\alpha)}+\tfrac{\lambda}{(1-\mu\eta^{\alpha-2})}\right)\delta<\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon,
\end{aligned}
$$

where

$$I=\frac{1}{\alpha-1}(t_2^{\alpha-1}-t_1^{\alpha-1})-\frac{1}{\alpha}\left(t_2{}^{\alpha}-t_1{}^{\alpha}\right).$$

By means of the Arzela-Ascoli theorem, $T:\mathcal{K}\longrightarrow\mathcal{K}$ is completely continuous. The proof is achieved. ■

In all what follow, we assume that the next conditions are satisfied.

(C1) $f:[0,\infty)\longrightarrow[0,\infty)$ is continuous;

(C2) $a:(0,1)\longrightarrow[0,\infty)$ is continuous, $0<\displaystyle\int_0^1 s(1-s)^{\alpha-2}a(s)ds<\infty$ and $0<\displaystyle\int_0^1(1-s)^{\alpha-2}a(s)ds<\infty$.(that is a is singular at $t=0$, $t=1$)

Lemma 12 *[15] Suppose that $\mathcal{E}$ is a Banach space, $T_n:\mathcal{E}\longrightarrow\mathcal{E}$ $(n=1,2,3,...)$ are completely continuous operators, $T:\mathcal{E}\longrightarrow\mathcal{E}$, and*

$$\lim_{n\longrightarrow\infty}\max_{\|u\|<r}\|T_nu-Tu\|=0\qquad \textit{for all}\quad r>0.$$

Then T is completely continuous.

For any natural number n $(n\geq 2)$, we set

$$a_n(t)=\begin{cases}\inf_{t<s\leq\frac{1}{n}}a(s), & 0\leq t\leq\frac{1}{n},\\ a(t), & \frac{1}{n}\leq t\leq 1-\frac{1}{n},\\ \inf_{1-\frac{1}{n}<s<t}a(s), & 1-\frac{1}{n}\leq t\leq 1.\end{cases}\tag{3.3}$$

Then $a_n:[0,1]\longrightarrow[0,+\infty)$ is continuous and $a_n(t)\leq a(t)$, $t\in(0,1)$. Let

$$T_nu(t)=\int_0^1G(t,s)a_n(s)f(u(s))ds+\tfrac{\mu t^{\alpha-1}}{(1-\mu\eta^{\alpha-2})}\int_0^1G_1(\eta,s)a_n(s)f(u(s))ds$$
$$+\tfrac{\lambda t^{\alpha-1}}{(\alpha-1)(1-\mu\eta^{\alpha-2})}.$$

Lemma 13 *If (C1), (C2) hold. Then $T:\mathcal{K}\longrightarrow\mathcal{K}$ is completely continuous.*

Proof. By a similar as in the proof of Lemma 11 it is obvious that $T_n:\mathcal{E}\longrightarrow\mathcal{E}$ is completely continuous.

Since

$$\begin{aligned}0&<\int_0^1G(t,s)a(s)ds+\tfrac{\mu t^{\alpha-1}}{(1-\mu\eta^{\alpha-2})}\int_0^1G_1(\eta,s)a(s)ds\\&<\int_0^1G(1,s)a(s)ds+\tfrac{\mu}{(1-\mu\eta^{\alpha-2})}\int_0^1G_1(\eta,s)a(s)ds\\&<\frac{1}{\Gamma(\alpha)}\left[\int_0^1s(1-s)^{\alpha-2}a(s)ds+\tfrac{\mu\eta^{\alpha-2}}{(1-\mu\eta^{\alpha-2})}\int_0^1(1-s)^{\alpha-2}a(s)ds\right]\\&<+\infty,\end{aligned}$$

and by the absolute continuity of the integral, we have

$$\lim_{n\longrightarrow\infty}\left[\int_{e(n)}s(1-s)^{\alpha-2}a(s)ds+\tfrac{\mu\eta^{\alpha-2}t^{\alpha-1}}{(1-\mu\eta^{\alpha-2})}\int_{e(n)}(1-s)^{\alpha-2}a(s)ds\right]=0,$$

where $e(n) = \left[0, \frac{1}{n}\right] \cup \left[1 - \frac{1}{n}, 1\right]$.

Let $r > 0$ and $u \in B_r = \{u \in \mathcal{E} : \|u\| \leq r\}$ and $M_r = \max\{f(u(t) : (t,u) \in [0,1] \times [0,r]\} < +\infty$, by (3.3), Lemma 6(P3), and the absolute continuity of the integral, we have

$$\begin{aligned}
&\lim_{n \longrightarrow \infty} \|T_n u - Tu\| \\
&\leq \lim_{n \longrightarrow \infty} \max_{0 \leq t < 1} \left| \begin{array}{c} \int_0^1 G(t,s)\,(a_n(s) - a(s))\, f(u(s))ds \\ + \frac{\mu t^{\alpha-1}}{(1-\mu\eta^{\alpha-2})} \int_0^1 G_1(\eta,s)\,(a_n(s) - a(s))\, f(u(s))ds \end{array} \right| \\
&\leq \frac{M_r}{\Gamma(\alpha)} \lim_{n \longrightarrow \infty} \left[\begin{array}{c} \int_0^1 s(1-s)^{\alpha-2}\,(a(s) - a_n(s))\,ds \\ + \frac{\mu}{(1-\mu\eta^{\alpha-2})} \int_0^1 (1-s)^{\alpha-2}\,(a(s) - a_n(s))\,ds \end{array} \right] \\
&\leq \frac{M_r}{\Gamma(\alpha)} \lim_{n \longrightarrow \infty} \left[\begin{array}{c} \int_{e(n)} s(1-s)^{\alpha-2}\,(a(s) - a_n(s))\,ds \\ + \frac{\mu}{(1-\mu\eta^{\alpha-2})} \int_{e(n)} (1-s)^{\alpha-2}\,(a(s) - a_n(s))\,ds \end{array} \right] \\
&= \frac{M_r}{\Gamma(\alpha)} \lim_{n \longrightarrow \infty} \left[\begin{array}{c} \int_0^{\frac{1}{n}} s(1-s)^{\alpha-2}\,(a(s) - a_n(s))\,ds \\ + \frac{\mu}{(1-\mu\eta^{\alpha-2})} \int_0^{\frac{1}{n}} (1-s)^{\alpha-2}\,(a(s) - a_n(s))\,ds \\ + \int_{1-\frac{1}{n}}^1 s(1-s)^{\alpha-2}\,(a(s) - a_n(s))\,ds \\ + \frac{\mu}{(1-\mu\eta^{\alpha-2})} \int_{1-\frac{1}{n}}^1 (1-s)^{\alpha-2}\,(a(s) - a_n(s))\,ds \end{array} \right] \\
&\leq \frac{M_r}{\Gamma(\alpha)} \lim_{n \longrightarrow \infty} \left[\int_{e(n)} s(1-s)^{\alpha-2} a(s)ds + \frac{\mu\eta^{\alpha-2} t^{\alpha-1}}{(1-\mu\eta^{\alpha-2})} \int_{e(n)} (1-s)^{\alpha-2}\, a(s)ds \right] = 0.
\end{aligned}$$

Then by Lemma 12, $T : \mathcal{K} \longrightarrow \mathcal{K}$ is completely continuous. ■

Throughout this section, we shall use the following notations:

$$\Lambda_1 := \left(\frac{1}{\Gamma(\alpha)} \int_0^1 s(1-s)^{\alpha-2} a(s)ds + \frac{\mu}{(1-\mu\eta^{\alpha-2})} \int_0^1 G_1(\eta,s) a(s)ds \right)^{-1},$$

$$\Lambda_2 := \left(\frac{\gamma}{\Gamma(\alpha)} \int_\tau^1 s(1-s)^{\alpha-2} a(s)ds + \frac{\mu\gamma}{(1-\mu\eta^{\alpha-2})} \int_\tau^1 G_1(\eta,s) a(s)ds \right)^{-1}.$$

It is obvious that $\Lambda_2 > \Lambda_1 > 0$. Also we define

$$f_0 = \lim_{r \longrightarrow 0^+} \frac{f(r)}{r},\ f_\infty = \lim_{r \longrightarrow \infty} \frac{f(r)}{r}.$$

Theorem 14 *Suppose that f is superlinear, i.e.*

$$f_0 = 0, \qquad f_\infty = \infty.$$

Then BVP (1.1)-(1.2) has at least one positive solution for λ small enough and has no positive solution for λ large enough.

Proof. We divide the proof into two steps.

Step 1. We prove that BVP (1.1)-(1.2) has at least one positive solution for sufficiently small $\lambda > 0$.since $f_0 = 0$, for $\Lambda_1 > 0$, there exists $R_1 > 0$ such that $\frac{f(r)}{r} \leq \frac{\Lambda_1}{2}$, $r \in [0, R_1]$. Therefore,

$$f(r) \leq \frac{r\Lambda_1}{2}, \text{ for } r \in [0, R_1]. \tag{3.4}$$

Let $\Omega_1 = \{u \in C[0,1] : \|u\| \le R_1\}$ and let λ satisfies

$$0 < \lambda \le \frac{(\alpha-1)\left(1-\mu\eta^{\alpha-2}\right)R_1}{2}. \tag{3.5}$$

Then, for any $u \in \mathcal{K} \cap \partial\Omega_1$, it follows from Lemma 6, (3.2), (3.4) and (3.5) that

$$\begin{aligned}
Tu(t) &= \int_0^1 G(t,s)a(s)f(u(s))ds + \tfrac{t^{\alpha-1}}{(1-\mu\eta^{\alpha-2})}\left(\mu\int_0^1 G_1(\eta,s)a(s)f(u(s))ds + \tfrac{\lambda}{(\alpha-1)}\right)\\
&\le \frac{1}{\Gamma(\alpha)}\int_0^1 s\,(1-s)^{\alpha-2}\,a(s)f(u(s))ds + \tfrac{\mu}{(1-\mu\eta^{\alpha-2})}\int_0^1 G_1(\eta,s)a(s)f(u(s))ds\\
&+ \tfrac{\lambda}{(\alpha-1)(1-\mu\eta^{\alpha-2})}\\
&\le \frac{\Lambda_1}{2}\left(\frac{1}{\Gamma(\alpha)}\int_0^1 s\,(1-s)^{\alpha-2}\,a(s)u(s)ds + \tfrac{\mu}{(1-\mu\eta^{\alpha-2})}\int_0^1 G_1(\eta,s)a(s)u(s)ds\right)\\
&+ \tfrac{(\alpha-1)\left(1-\mu\eta^{\alpha-2}\right)R_1}{2(\alpha-1)(1-\mu\eta^{\alpha-2})}\\
&\le \frac{\Lambda_1}{2}\left(\frac{1}{\Gamma(\alpha)}\int_0^1 s\,(1-s)^{\alpha-2}\,a(s)ds + \tfrac{\mu}{(1-\mu\eta^{\alpha-2})}\int_0^1 G_1(\eta,s)a(s)ds\right)\|u\| + \tfrac{R_1}{2}\\
&= \tfrac{\|u\|}{2} + \tfrac{\|u\|}{2} = \|u\|.
\end{aligned}$$

And thus

$$\|Tu(t)\| \le \|u\|, \quad \text{for } u \in K \cap \partial\Omega_1. \tag{3.6}$$

On the other hand, since $f_\infty = \infty$, for $\Lambda_2 > 0$, there exists $R_2 > R_1$ such that $\frac{f(r)}{r} \ge \Lambda_2$, $r \in [\gamma R_2, \infty)$ .Thus we have

$$f(r) \ge r\Lambda_2, \text{ for } r \in [\gamma R_2, \infty]. \tag{3.7}$$

Set $\Omega_2 = \{u \in C[0,1] : \|u\| \le R_2\}$. For any $u \in \mathcal{K} \cap \partial\Omega_2$, by Lemma 6 one has $\min_{s\in[\tau,1]} u(s) \ge \gamma\|u\| = \gamma R_2$. Thus , from (3.6) we can conclude that

$$\begin{aligned}
Tu(1) &= \int_0^1 G(1,s)a(s)f(u(s))ds + \tfrac{\mu}{(1-\mu\eta^{\alpha-2})}\int_0^1 G_1(\eta,s)a(s)f(u(s))ds\\
&+ \tfrac{\lambda}{(\alpha-1)(1-\mu\eta^{\alpha-2})}\\
&\ge \int_0^1 G(1,s)a(s)f(u(s))ds + \tfrac{\mu}{(1-\mu\eta^{\alpha-2})}\int_0^1 G_1(\eta,s)a(s)f(u(s))ds\\
&\ge \frac{1}{\Gamma(\alpha)}\int_\tau^1 (1-s)^{\alpha-2} s a(s)f(u(s))ds + \tfrac{\mu}{(1-\mu\eta^{\alpha-2})}\int_\tau^1 G_1(\eta,s)a(s)f(u(s))ds\\
&\ge \Lambda_2\left(\frac{1}{\Gamma(\alpha)}\int_\tau^1 (1-s)^{\alpha-2} s a(s)u(s)ds + \tfrac{\mu}{(1-\mu\eta^{\alpha-2})}\int_\tau^1 G_1(\eta,s)a(s)u(s)ds\right)\\
&\ge \Lambda_2\left(\frac{\gamma}{\Gamma(\alpha)}\int_\tau^1 (1-s)^{\alpha-2} s a(s)ds + \tfrac{\mu\gamma}{(1-\mu\eta^{\alpha-2})}\int_\tau^1 G_1(\eta,s)a(s)ds\right)\|u\|\\
&= \|u\|,
\end{aligned}$$

which implies that

$$\|Tu\| \ge \|u\|, \quad \text{for } u \in \mathcal{K} \cap \partial\Omega_2. \tag{3.8}$$

Therefore, by (3.6), (3.8) and the first part of Theorem 10 we know that the operator T has at least one fixed point $u \in \mathcal{K} \cap \left(\overline{\Omega}_2 \backslash \Omega_1\right)$, which is a positive solution of BVP (1.1)-(1.2).

Step 2. We verify that BVP (1.1)-(1.2) has no positive solution for λ large enough. Otherwise, there exist $0 < \lambda_1 < \lambda_2 < ... < \lambda_n < ...$, with $\lim_{n \longrightarrow \infty} \lambda_n = +\infty$, such that for any positive integer n, the BVP

$$\begin{cases} D^\alpha u + a(t) f(u) = 0, \ 2 < \alpha < 3, \ 0 < t < 1, \\ \quad u(0) = u^{'}(0) = 0, \ \ u^{'}(1) - \mu u^{'}(\eta) = \lambda_n, \end{cases}$$

has a positive solution $u_n(t)$, by (3.1), we have

$$\begin{aligned} u_n(1) &= \int_0^1 G(1,s) a(s) f(u_n(s)) ds \\ &+ \frac{1}{(1-\mu\eta^{\alpha-2})} \left(\mu \int_0^1 G_1(\eta, s) a(s) f(u_n(s)) ds + \frac{\lambda_n}{(\alpha-1)} \right) \\ &\geq \frac{\lambda_n}{(\alpha-1)(1-\mu\eta^{\alpha-2})} \longrightarrow +\infty, \quad (n \longrightarrow \infty). \end{aligned}$$

Thus

$$\|u\| \longrightarrow +\infty, \quad (n \longrightarrow \infty).$$

Since $f_\infty = \infty$, for $4\Lambda_2 > 0$, there exists $\widehat{R} > 0$ such that $\frac{f(r)}{r} \geq 4\Lambda_2$, $r \in \left[\gamma \widehat{R}, \infty\right)$, which implies that

$$f(r) \geq 2\Lambda_2 r, \text{ for } r \in \left[\gamma \widehat{R}, \infty\right).$$

Let n be large enough that $\|u_n\| \geq \widehat{R}$, then

$$\begin{aligned} \|u_n\| &\geq u_n(1) \\ &= \int_0^1 G(1,s) a(s) f(u_n(s)) ds + \tfrac{\mu}{(1-\mu\eta^{\alpha-2})} \int_0^1 G_1(\eta,s) a(s) f(u_n(s)) ds \\ &+ \tfrac{\lambda_n}{(\alpha-1)(1-\mu\eta^{\alpha-2})} \\ &\geq 2\Lambda \left(\int_0^1 G(1,s) a(s) u_n(s) ds + \tfrac{\mu}{(1-\mu\eta^{\alpha-2})} \int_0^1 G_1(\eta,s) a(s) u_n(s) ds \right) \\ &\geq 2\Lambda \left(\int_\tau^1 G(1,s) a(s) u_n(s) ds + \tfrac{\mu}{(1-\mu\eta^{\alpha-2})} \int_\tau^1 G_1(\eta,s) a(s) u_n(s) ds \right) \\ &\geq 2\Lambda_2 \left(\gamma \frac{1}{\Gamma(\alpha)} \int_\tau^1 (1-s)^{\alpha-2} s a(s) ds + \tfrac{\mu\gamma}{(1-\mu\eta^{\alpha-2})} \int_\tau^1 G_1(\eta,s) a(s) ds \right) \|u_n\| \\ &= 2 \|u_n\|, \end{aligned}$$

which is contradiction. The proof is complete. ■

Moreover, if the function f is nondecreasing, the following theorem holds.

Theorem 15 *Suppose that f is superlinear. If f is nondecreasing, then there exists a positive constant λ^* such that BVP (1.1)-(1.2) has at least one positive solution for $\lambda \in (0, \lambda^*)$ and has no positive solution for $\lambda \in (\lambda^*, \infty)$.*

Proof. Let $\Sigma = \{\lambda : \text{BVP (1.1)-(1.2) has at least one positive solution}\}$ and $\lambda^* = \sup \Sigma$; it follows from Theorem 14 that $0 < \lambda^* < \infty$. From the definition of λ^*, we know that for any $\lambda \in (0, \lambda^*)$, there is a $\lambda_0 > \lambda$ such that BVP

$$\begin{cases} D^\alpha u + a(t) f(u(t)) = 0,\ 2 < \alpha < 3,\ 0 < t < 1, \\ u(0) = u'(0) = 0,\ \ u'(1) - \mu u'(\eta) = \lambda_0, \end{cases}$$

has a positive solution $u_0(t)$. Now we prove that for any $\lambda \in (0, \lambda_0)$, BVP (1.1)-(1.2) has a positive solution. In fact, let

$$\mathcal{K}(u_0) = \{u \in \mathcal{K} : u(t) \leq u_0(t),\ t \in [0,1]\}$$

For any $\lambda \in (0, \lambda_0)$, $u \in \mathcal{K}(u_0)$, it follows from (3.2) and the monotonicity of f that we have that

$$\begin{aligned} Tu(t) &= \int_0^1 G(t,s)a(s)f(u(s))ds + \tfrac{t^{\alpha-1}}{(1-\mu\eta^{\alpha-2})}\left(\mu \int_0^1 G_1(\eta,s)a(s)f(u(s))ds + \tfrac{\lambda}{(\alpha-1)}\right) \\ &\leq \int_0^1 G(t,s)a(s)f(u_0(s))ds + \tfrac{t^{\alpha-1}}{(1-\mu\eta^{\alpha-2})}\left(\mu \int_0^1 G_1(\eta,s)a(s)f(u_0(s))ds + \tfrac{\lambda}{(\alpha-1)}\right) \\ &= u_0(t). \end{aligned}$$

Thus, $T(\mathcal{K}(u_0)) \subseteq \mathcal{K}(u_0)$. By Shaulder's fixed point theorem we know that there exists a fixed point $u \in \mathcal{K}(u_0)$, which is a positive solution of BVP (1.1)-(1.2). The proof is complete. ■

Now we consider the case f is sublinear.

Theorem 16 *Suppose that f is sublinear, i.e.*

$$f_0 = \infty,\ \ f_\infty = 0.$$

Then BVP (1.1)-(1.2) has at least one positive solution for any $\lambda \in (0, \infty)$.

Proof. Since $f_0 = \infty$, there exists $R_1 > 0$ such that $f(r) \geq \Lambda_2 r$, for any $r \in [0, R_1]$. So for any $u \in \mathcal{K}$ with $\|u\| = R_1$ and any $\lambda > 0$, we have

$$\begin{aligned} Tu(1) &= \int_0^1 G(1,s)a(s)f(u(s))ds + \tfrac{1}{(1-\mu\eta^{\alpha-2})}\left(\mu \int_0^1 G_1(\eta,s)a(s)f(u(s))ds + \tfrac{\lambda}{(\alpha-1)}\right) \\ &\geq \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-2} s a(s) f(u(s))ds + \tfrac{\mu}{(1-\mu\eta^{\alpha-2})} \int_0^1 G_1(\eta,s)a(s)f(u(s))ds \\ &\geq \frac{1}{\Gamma(\alpha)} \int_\tau^1 (1-s)^{\alpha-2} s a(s) f(u(s))ds + \tfrac{\mu}{(1-\mu\eta^{\alpha-2})} \int_\tau^1 G_1(\eta,s)a(s)f(u(s))ds \\ &\geq \Lambda_2 \left(\frac{1}{\Gamma(\alpha)} \int_\tau^1 (1-s)^{\alpha-2} s a(s) u(s)ds + \tfrac{\mu}{(1-\mu\eta^{\alpha-2})} \int_\tau^1 G_1(\eta,s)a(s)u(s)ds\right) \\ &\geq \Lambda_2 \left(\gamma \frac{1}{\Gamma(\alpha)} \int_\tau^1 (1-s)^{\alpha-2} s a(s)ds + \tfrac{\mu\gamma}{(1-\mu\eta^{\alpha-2})} \int_\tau^1 G_1(\eta,s)a(s)ds\right) \|u\| \\ &= \|u\|, \end{aligned}$$

and consequently, $\|Tu\| \geq \|u\|$. So, if we set $\Omega_1 = \{u \in \mathcal{K} : \|u\| < R_1\}$, then

$$\|Tu\| \geq \|u\|, \text{ for } u \in \mathcal{K} \cap \partial\Omega_1. \tag{3.9}$$

Next we construct the set Ω_2. We consider two cases: f is bounded or f is unbounded.

Case (1): Suppose that f is bounded, say $f(r) \leq M$ for all $r \in [0, \infty)$. In this case we choose

$$R_2 \geq \max\left\{2R_1, \frac{2M}{\Lambda_1}, \frac{2\lambda}{(\alpha-1)(1-\mu\eta^{\alpha-2})}\right\},$$

and then for $u \in \mathcal{K}$ with $\|u\| = R_2$, we have

$$\begin{aligned} Tu(t) &= \int_0^1 G(t,s)a(s)f(u(s))ds + \tfrac{t^{\alpha-1}}{(1-\mu\eta^{\alpha-2})}\left(\mu\int_0^1 G_1(\eta,s)a(s)f(u(s))ds + \tfrac{\lambda}{(\alpha-1)}\right) \\ &\leq M\left(\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-2}sa(s)ds + \tfrac{\mu}{(1-\mu\eta^{\alpha-2})}\int_0^1 G_1(\eta,s)a(s)ds\right) \\ &+ \tfrac{\lambda}{(\alpha-1)(1-\mu\eta^{\alpha-2})} \\ &\leq \frac{M}{\Lambda_1} + \frac{R_2}{2} \leq \frac{R_2}{2} + \frac{R_2}{2} = R_2 = \|u\|. \end{aligned}$$

So,

$$\|Tu\| \leq \|u\|.$$

Case (2): When f is unbounded. Now, since $f_\infty = 0$, there exists $R_0 > 0$ such that

$$f(r) \leq \frac{\Lambda_1}{2}r, \text{ for } r \in [R_0, \infty), \tag{3.10}$$

Let

$$R_2 \geq \max\left\{2R_1, R_0, \frac{2\lambda}{(\alpha-1)(1-\mu\eta^{\alpha-2})}\right\},$$

and be such that

$$f(r) \leq f(R_2), \text{ for } r \in [0, R_2].$$

For $u \in \mathcal{K}$ with $\|u\| = R_2$, from (3.2) and (3.10), we have

$$\begin{aligned} Tu(t) &= \int_0^1 G(t,s)a(s)f(u(s))ds + \tfrac{t^{\alpha-1}}{(1-\mu\eta^{\alpha-2})}\left(\mu\int_0^1 G_1(\eta,s)a(s)f(u(s))ds + \tfrac{\lambda}{(\alpha-1)}\right) \\ &\leq \int_0^1 G(t,s)a(s)f(R_2)ds + \tfrac{1}{(1-\mu\eta^{\alpha-2})}\left(\mu\int_0^1 G_1(\eta,s)a(s)f(R_2)ds + \tfrac{\lambda}{(\alpha-1)}\right) \\ &\leq \frac{\Lambda_1}{2}\left(\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-2}sa(s)ds + \tfrac{\mu}{(1-\mu\eta^{\alpha-2})}\int_0^1 G_1(\eta,s)a(s)ds\right)R_2 + \frac{R_2}{2} \\ &= \frac{R_2}{2} + \frac{R_2}{2} = R_2 = \|u\|. \end{aligned}$$

Thus,

$$\|Tu\| \leq \|u\|.$$

Therefore, in either case we may put $\Omega_2 = \{u \in \mathcal{K} : \|u\| < R_2\}$, then

$$\|Tu\| \geq \|u\|, \text{ for } u \in \mathcal{K} \cap \partial\Omega_2. \tag{3.11}$$

So, it follows from (3.9), (3.11) and the second part of Theorem 10 that T has a fixed point $u^* \in \mathcal{K} \cap \left(\overline{\Omega}_2 \backslash \Omega_1\right)$, Then u^* is a positive solution of BVP (1.1)-(1.2). The proof is achieved. ■

4 Application

In this section we give an example to illustrate the usefulness of our main results.

Example 17 *Let us consider the following fractional BVP*

$$D_{0^+}^{\frac{5}{2}}u(t)+\frac{1}{\sqrt{t\,(1-t^2)}}u^{\frac{3}{2}}(t)=0,\ 0<t<1, \tag{4.1}$$

$$u(0)=u'(0)=0,\ \ u'(1)-\frac{1}{2\sqrt{2}}u'(\frac{1}{2})=\lambda, \tag{4.2}$$

We can easily show that $f(u(t))=u^{\frac{3}{2}}(t)$ satisfy:

$$f_0=\lim_{u\longrightarrow 0^+}\frac{f(u)}{u}=\lim_{u\to 0^+}\sqrt{u(t)}=0,\quad f_\infty=\lim_{u\to\infty}\frac{f(u)}{u}=\lim_{u\to\infty}\sqrt{u(t)}=+\infty,$$

obviously, for a.e. $t\in[0,1]$, we have

$$\int_0^1(1-s)^{\alpha-2}a(s)ds=\int_0^1\frac{\sqrt{1-s}}{\sqrt{s\,(1-s^2)}}ds=\int_0^1\frac{ds}{\sqrt{s\,(1+s)}}=1.762\,7.$$
$$\int_0^1 s(1-s)^{\alpha-2}a(s)ds=\int_0^1\frac{s\sqrt{1-s}}{\sqrt{s\,(1-s^2)}}ds=\int_0^1\sqrt{\frac{s}{1+s}}ds=0.532\,84.$$

So conditions (C1), (C2) holds, then we can choose $R_2>R_1>0$, and for λ satisfies $0<\lambda\leq\frac{9}{16}R_1<R_2$, then we can choose

$$\Omega_1=\{u\in\mathcal{K}:\|u\|<R_1\},\ \Omega_2=\{u\in\mathcal{K}:\|u\|<R_2\}$$

and by Theorem 14, we can show that the BVP (4.1)-(4.2) has at least one positive solution $u(t)\in\mathcal{K}\cap\left(\overline{\Omega}_2\backslash\Omega_1\right)$ for λ small enough and has no positive solution for λ large enough.

Example 18 *Let us consider the following fractional BVP*

$$D_{0^+}^{\frac{5}{2}}u(t)+\frac{1}{\sqrt{t\,(1-t^2)}}\exp(-u(t))=0,\ 0<t<1, \tag{4.3}$$

$$u(0)=u'(0)=0,\ \ u'(1)-\frac{1}{2\sqrt{2}}u'(\frac{1}{2})=\lambda, \tag{4.4}$$

We can easily show that $f(u(t))=\exp(-u(t))$ satisfy:

$$f_0=\lim_{u\longrightarrow 0^+}\frac{f(u)}{u}=\lim_{u\to 0^+}\frac{1}{u\exp(u)}=\infty,\ f_\infty=\lim_{u\to\infty}\frac{f(u)}{u}=\lim_{u\to\infty}\frac{1}{u\exp(u)}=0,$$

obviously, for a.e. $t \in [0,1]$, we have

$$\int_0^1 (1-s)^{\alpha-2} a(s) ds = \int_0^1 \frac{ds}{\sqrt{s(1+s)}} = 1.7627.$$
$$\int_0^1 s(1-s)^{\alpha-2} a(s) ds = \int_0^1 \sqrt{\frac{s}{1+s}} ds = 0.53284.$$

So conditions (C1), (C2) holds, and by Theorem 16, we can show that the BVP (4.3)-(4.4) has at least one positive solutions $u(t)$, for any $\lambda \in (0,\infty)$.

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